

A New Proof for Classification of Irreducible Modules of a Hecke Algebra of Type A_n

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Dedicated to Professor Gus Lehrer on his sixtieth birthday

ABSTRACT. In this paper we give a new proof for the classification of irreducible modules of an affine Hecke algebra of type A_n , which was obtained by G. E. Murphy in 1995.

Let H be the Hecke algebra of the symmetric group S_n over a commutative ring K with an invertible parameter $q \in K$. In [M] Murphy worked out a classification of irreducible modules of H when K is a field, which is similar to the classification of irreducible modules of a symmetric group over a field [J]. In this paper we shall give a new proof for Murphy's classification. Essentially the idea is due to Murphy, but we use Kazhdan-Lusztig theory and affine Hecke algebra of type \tilde{A}_{n-1} to prove his result by a direct calculation.

As usual, the simple reflections of S_n consisting of the transposes $s_i = (i, i+1)$ for $i = 1, 2, \dots, n-1$. As a free K -module, the Hecke algebra H has a basis T_w , $w \in S_n$, and the multiplication is defined by the relations $(T_s - q)(T_s + 1) = 0$ if s is a simple reflection, $T_w T_u = T_{wu}$ if $l(wu) = l(w) + l(u)$, here $l: S_n \rightarrow \mathbf{N}$ is the length function.

For each partition $\lambda = (\lambda_1, \dots, \lambda_k)$ of n , set $I_j = \{\lambda_1 + \dots + \lambda_{j-1} + 1, \lambda_1 + \dots + \lambda_{j-1} + 2, \dots, \lambda_1 + \dots + \lambda_{j-1} + \lambda_j\}$ for $1 \leq j \leq k$ (we understand $\lambda_{-1} = 0$). Let S_λ be the subgroup of S_n consisting of elements stabilizing each I_j . Then S_λ is a parabolic subgroup of S_n .

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and is isomorphic to $S_{\lambda_1} \times S_{\lambda_2} \times \cdots \times S_{\lambda_k}$. We shall denote by w_λ the longest element of S_λ . Set $C_\lambda = \sum_{w \in S_\lambda} T_w$. Following [KL] and [M] we consider the left idea $N_\lambda = HC_\lambda$ of H and shall regard it as a left H -module. Let N'_λ be the maximal submodule of N_λ not containing C_λ . Then the quotient module $M_\lambda = N_\lambda/N'_\lambda$ is an irreducible module of H . Assume that K is a field, then each irreducible module of H is isomorphic to some M_λ . See [KL, proof of Theorem 1.4] or [M]. When $\sum_{w \in S_n} q^{l(w)} \neq 0$, the irreducible modules M_λ , λ a partition of n , form a complete set of irreducible modules of H (see [G, M], when q is not a root of 1, this result was implied in [L]).

One of the main result in [M] is the following.

Theorem. Assume that K is a field. Then

- (a) The set $\{M_\lambda \mid C_\lambda M_\lambda \neq 0\}$ is a complete set of irreducible modules of H .
- (b) $C_\lambda M_\lambda \neq 0$ if and only if $\sum_{a=0}^m q^a \neq 0$ for all $1 \leq m \leq \max\{\lambda_1 - \lambda_2, \lambda_2 - \lambda_3, \dots, \lambda_{k-1} - \lambda_k, \lambda_k\}$. (See [M, Theorems 6.4 and 6.9]).

Now we argue for the theorem. For each module E we can attached a partition $\lambda = p(E)$ as follows, $C_\lambda E \neq 0$ but $C_\mu E = 0$ for all partition μ satisfying $\mu > \lambda$. (We say that $\mu = (\mu_1, \mu_2, \dots, \mu_j) \geq \lambda = (\lambda_1, \lambda_2, \dots, \lambda_k)$ if $\mu_1 + \cdots + \mu_i \geq \lambda_1 + \cdots + \lambda_i$ for $i = 1, 2, \dots$)

Consider the two-sided ideal $F_\lambda = HC_\lambda H$ of H . According to the proof of Theorem 1.4 in [KL], $F_\lambda / (F_\lambda \cap \sum_{\mu > \lambda} F_\mu)$ is isomorphic to the direct sum of some copies of $E_\lambda = N_\lambda / (N_\lambda \cap \sum_{\mu > \lambda} F_\mu)$.

Let E'_λ be the sum of all submodules E of E_λ satisfying $C_\lambda E = 0$. We claim that either $E'_\lambda = E_\lambda$ or E'_λ is the unique maximal submodule of E_λ .

Let D be a submodule of E_λ such that $C_\lambda D \neq 0$. For any $h \in H$ we have $C_\lambda h C_\lambda \in a C_\lambda + \sum_{\mu > \lambda} F_\mu$, here $a \in K$ (loc.cit). Thus $C_\lambda D \neq 0$ implies that $C_\lambda D = E_\lambda$. Therefore $E'_\lambda = E_\lambda$ or E'_λ is the unique maximal submodule of E_λ . As a consequence, $M_\lambda = E_\lambda / E'_\lambda$ if $C_\lambda E_\lambda \neq 0$ and in this case $C_\lambda M_\lambda \neq 0$.

Now assume that L is an irreducible H -module such that $C_\lambda L \neq 0$ but $C_\mu L = 0$ for all $\mu > \lambda$. Let $x \in L$ be such that $C_\lambda x \neq 0$. Consider

the H -module homomorphism $N_\lambda \rightarrow L$, $C_\lambda \rightarrow C_\lambda x$. By assumption, $F_\mu L = 0$ if $\mu > \lambda$. Thus we get a nonzero homomorphism $E_\lambda \rightarrow L$. We must have $C_\lambda E_\lambda \neq 0$ since $C_\lambda L \neq 0$. So L is isomorphic to M_λ . Noting that $C_\mu E_\lambda \neq 0$ implies that $\mu \leq \lambda$ (loc.cit) we see that if $\lambda \neq \mu$ then M_λ is not isomorphic to M_μ when $C_\lambda M_\lambda \neq 0 \neq C_\mu M_\mu$. Part (a) is proved.

To prove part (b) we need calculate $C_\lambda H C_\lambda$. This is equivalent to calculate all $C_\lambda T_w C_\lambda$. Clearly if $w \in S_\lambda$, then $T_w C_\lambda = q^{l(w)} C_\lambda$. So we only need consider the element of minimal length in a double coset $S_\lambda w S_\lambda$. Now the affine Hecke algebra plays a role in calculating the product $C_\lambda T_w C_\lambda$.

Let G be the special linear group $SL_n(\mathbf{C})$ and let T be the subgroup of G consisting of diagonal matrices. Let $X = \text{Hom}(T, \mathbf{C}^*)$ be the character group of T . Let $\tau_i \in X$ be the character $T \rightarrow \mathbf{C}$, $\text{diag}(a_1, a_2, \dots, a_n) \rightarrow a_i$. Then we have $\tau_1 \tau_2 \cdots \tau_n = 1$ and as a free abelian group X is generated by τ_i , $i = 1, 2, \dots, n-1$. The symmetric group S_n acts on X naturally: $w : X \rightarrow X$, $\tau_i \rightarrow \tau_{w(i)}$. Thus we can form the semi-direct product $\tilde{S}_n = S \ltimes X$. In \tilde{S}_n we have $w\tau_i = \tau_{w(i)}w$ for w in S_n . Let $s_0 = s\tau_1^2\tau_2 \cdots \tau_i \cdots \tau_{n-1}$, where $s \in S_n$ is the transpose $(1, n) = s_1 s_2 \cdots s_{n-2} s_{n-1} s_{n-2} \cdots s_2 s_1$. Since $\tau_1 \tau_2 \cdots \tau_n = 1$ we have $s_0^2 = 1$. The simple reflections s_0, s_1, \dots, s_{n-1} generate a subgroup W of \tilde{S}_n , which is a Coxeter group of type \tilde{A}_{n-1} . Define $\omega = \tau_1 s_1 s_2 \cdots s_{n-1}$. Then $\omega^n = 1$ and $\omega s_i = s_{i+1} \omega$ for all i (we set $s_n = s_0$). Let Ω be the subgroup of \tilde{S}_n generated by ω . Note that W is a normal subgroup of \tilde{S}_n and we have $\tilde{S}_n = \Omega \ltimes W$. The Hecke algebra \tilde{H} of \tilde{S}_n is defined as follows. As a K -module, it is free and has a basis consisting of elements T_w , $w \in \tilde{S}_n$. The multiplication is defined by the relations $(T_{s_i} - q)(T_{s_i} + 1) = 0$ for all i and $T_w T_u = T_{wu}$ if $l(wu) = l(w) + l(u)$. The length function $l : \tilde{S}_n \rightarrow \mathbf{N}$ is defined as $l(\omega^a w) = l(w)$ for $w \in W$. Clearly H is a subalgebra of \tilde{H} .

For $1 \leq i \leq n-1$, define $x_i = \tau_1 \tau_2 \cdots \tau_i$. Then we have $s_i x_j = x_j s_i$ if i and j are different. Moreover we have $l(w_0 \prod_{i=1}^{n-1} x_i^{a_i}) = l(w_0) + \sum_{i=1}^{n-1} a_i l(x_i)$ if all a_i are non-negative integers. Here w_0 is the longest element of S_n . Also we have $l(x_i s_j) = l(x_i) - 1$ if and only if $i = j$.

Thus we have $T_{s_i}T_{x_j} = T_{x_j}T_{s_i}$ if $1 \leq i \neq j \leq n-1$ and $T_{x_i} = T_{x_i s_i}T_{s_i}$.

For a positive integer k we set $[k] = q^{k-1} + q^{k-2} + \cdots + q + 1$, $[k]! = [k][k-1] \cdots [2][1]$, we also set $[0] = [0]! = 1$. For any element $w \in \tilde{S}_n$ we set $C_w = \sum_{y \leq w} P_{y,w}(q)T_y$, where \leq is the Bruhat order and $P_{y,w}$ is the Kazhdan-Lusztig polynomial. Note that if w is a longest element of a parabolic subgroup of \tilde{S}_n , then $C_w = \sum_{y \leq w} T_y$. So we have $C_\lambda = C_{w_\lambda}$. Now we are ready to prove part (b) of the theorem.

Lemma 1. Let $\lambda = (i, 1, \dots, 1)$ be a partition of n and $z \in S_n$ such that for any simple reflection s , $sz \leq z$ if and only if $s = s_i$ and $zs \leq z$ if and only if $s = s_i$. Then

$$C_\lambda T_z C_\lambda \in \pm q^*[i-j-1]! C_\mu + \sum_{\nu} F_\nu,$$

for some $j \leq i-1$, where $*$ stands for an integer, $\mu = (i, j+1, 1, \dots, 1)$, the summation runs through $\nu = (i+m, j+1-m, 1, \dots, 1) > \mu$ for $j+1 \geq m \geq 1$.

Proof: Since for any simple reflection s , if $sz \leq z$ or $zs \leq z$ then we have $s = s_i$, we can find $j \leq i-1$ such that

$$z = (s_i s_{i-1} \cdots s_{i-j})(s_{i+1} s_i \cdots s_{i-j+1}) \cdots (s_{i+j-1} s_{i+j-2} \cdots s_{i-1})(s_{i+j} s_{i+j-1} \cdots s_i).$$

It is no harm to assume $n = i + j + 1$.

Note that

$$x_i = \omega^i (s_{n-i} s_{n-i-1} \cdots s_1)(s_{n-i+1} s_{n-i} \cdots s_2) \cdots (s_{n-1} s_{n-2} \cdots s_i).$$

Let $y = (s_{i-j-1} s_{i-j} \cdots s_{i-1}) \cdots (s_2 s_3 \cdots s_{j+2})(s_1 s_2 \cdots s_{j+1})$. Since $n = i+j+1$ we have $z = y\omega^{-i}x_i$ and $l(x_i) = l(y^{-1}) + l(z)$ (we understand that $y = e$ if $j = i-1$.) Thus we have $C_\lambda T_z C_\lambda = C_\lambda T_{y^{-1}}^{-1} T_\omega^{-i} T_{x_i} C_\lambda$. Noting that $C_\lambda T_{y^{-1}}^{-1} = q^{-l(y)} C_\lambda$ and $C_\lambda T_{x_i} = T_{x_i} C_\lambda$, we get

$$C_\lambda T_z C_\lambda = q^{-l(y)} C_\lambda T_\omega^{-i} T_{x_i} C_\lambda = q^{-l(y)} T_\omega^{-i} T_\omega^i C_\lambda T_\omega^{-i} C_\lambda T_{x_i}.$$

Let $w' = \omega^i w_\lambda \omega^{-i}$. Then w' is the longest element of the subgroup of \tilde{S}_n generated by $s_{i+1}, s_{i+2}, \dots, s_{i+i-1}$. Let $k = i - j - 2$, then $2i - 1 = k + i + j + 1$. We have $w' = uw_k$ for some u and $l(w') = l(u) + l(w_k)$, where w_k is the longest element of the subgroup W_k of S_n generated by s_1, s_2, \dots, s_k if $k \geq 1$ and $w_k = e$ is the neutral element if $k = -1$ or 0 . We also have $u = u'u_{i+1}$ for some u' and $l(u) = l(u') + l(u_{i+1})$, where

u_{i+1} is the longest element of the subgroup of U_{i+1} of S_n generated by $s_{i+1}, \dots, s_{i+j} = s_{n-1}$. So $T_\omega^i C_\lambda T_\omega^{-i} = h C_{u_{i+1}} C_{w_k}$ for some h in H , where $C_{u_{i+1}}$ is the sum of all T_x , $x \in U_{i+1}$, and C_{w_k} is the sum of all T_x , $x \in W_k$. Clearly we have $C_{w_k} C_\lambda = [k+1]! C_\lambda$ and $C_{u_{i+1}} C_\lambda = C_\mu$. Therefore $C'_\lambda C_\lambda = [k+1]! h C_\mu$. Note that $uw_\lambda = u' u_{i+1} w_\lambda = u' w_\mu$ is in the subgroup of \tilde{S}_n generated by s_p , $p \neq i$. The subgroup is isomorphic to the symmetric group S_n . Applying the Robinson-Schensted rule we see that uw_λ and w_μ are in the same left cell. (See [A] for an exposition of Robinson-Schensted rule. One may see this fact also from star operations introduced in [KL].) Write $C'_\lambda C_\lambda = \sum a_v C_v$, then clearly $a_{uw_\lambda} = [k+1]!$. Since T_ω and T_{x_i} are invertible, we see that in the expression $C_\lambda T_z C_\lambda = \sum b_v C_v$, $b_v \in K$, there exists x such that $b_x \neq 0$, x and w_μ are in the same two-sided cell. Since $z = z^{-1}$ and $w_\lambda = w_\lambda^{-1}$, by the symmetry we see that x and w_μ are in the same left cell and right cell as well. So we must have $x = w_\mu$ (see [KL, proof of Theorem 1.4]). Moreover we must have $b_\mu = \pm q^a [k+1]!$ for some integer a . If $b_v \neq 0$ and $v \neq w_\mu$, we must have $C_v \in F_\nu$ for some $\nu > \mu$. We claim that for such ν we have $\nu = (i+m, j+1-m, 1, \dots, 1)$ for some $m \geq 1$. Since $C_\lambda T_z C_\lambda$ is contained in the subalgebra of H generated by $T_{s_1}, \dots, T_{s_{i+j}}$, we may assume that $n = i+j+1$. In this case we must have $\nu = (i+m, j+1-m)$ for some $m \geq 1$ since $\mu = (i, j+1)$ and $\nu > \mu$. The lemma is proved.

Remark: The author has not been able to determine the integer $*$ in the lemma.

Corollary 2. Let $\lambda = (i, j)$ be a partition of n . That is $i \geq j$ and $i+j = n$. Then for any z in S_n we have

$$C_\lambda T_z C_\lambda \in [i-j]! [j]! f C_\lambda + \sum_{\mu > \lambda} F_\mu,$$

where $f \in K$.

Proof: Since $C_\lambda C_\lambda = [i]! [j]! C_\lambda$ and $T_s C_\lambda = C_\lambda T_s = q C_\lambda$ if $s \neq i$ in S_n , we may assume that $z = (s_i s_{i-1} \cdots s_{i-k}) \cdots (s_{i+k} s_{i+k-1} \cdots s_i)$, where $k \leq j-1 \leq i-1$. Note that $C_\lambda = C_{w_{i-1}} C_{u_{i+1}} = C_{u_{i+1}} C_{w_{i-1}}$ (see the proof of Lemma 1 for the definition of w_i and u_i). We have

$C_\lambda T_z C_\lambda = C_{u_{i+1}} C_{w_{i-1}} T_z C_{w_{i-1}} C_{u_{i+1}}$. By Lemma 1 we get $C_{w_{i-1}} T_z C_{w_{i-1}} \in \pm q^*[i-k-1]! C_{w_{i-1} w_{i+1, i+k}} + \sum_\nu F_\nu$, where $w_{i+1, i+k}$ is the longest element of the subgroup of $W_{i+1, i+k}$ of S_n generated by s_{i+1}, \dots, s_{i+k} , and ν runs through the partitions $(i+m, k+1-m, 1, \dots, 1)$, $k+1 \geq m \geq 1$.

We have $C_{u_{i+1}} C_{w_{i+1, i+k}} C_{u_{i+1}} = [k+1]![j]! C_{u_{i+1}}$. We also have $C_\lambda T_z C_\lambda \subset \sum_{\mu \geq \lambda} F_\mu$ and $C_{u_{i+1}} F_\nu C_{u_{i+1}} \subset \sum_{\mu \geq \nu} F_\mu$ for any ν . If $\mu \geq \lambda$ and $\mu \geq (i+m, \dots)$ for some $m \geq 1$, we must have $\mu > \lambda$. So $C_\lambda T_z C_\lambda \in \pm q^*[i-k-1]![k+1]![j]! C_\lambda + \sum_{\mu > \lambda} F_\mu$. Since $[i-k-1]! = [i-j]![i-j+1] \cdots [i-k-1]$, the corollary follows.

Lemma 3. Let $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_k)$ be a partition of n . Then

$$C_\lambda T_z C_\lambda \in \prod_{i=1}^k [\lambda_i - \lambda_{i+1}]! f C_\lambda + F_{>\lambda},$$

where $f \in K$ and we set $\lambda_{k+1} = 0$.

Proof: We use induction on k . When $k = 1$, the lemma is trivial, when $k = 2$, by Corollary 2 we see the assertion is true. Now assume that $k > 2$. For $i \leq j$ we set $l_{i,j} = \lambda_i + \dots + \lambda_j$. We have (see the proof of Corollary 2 for the definition of w_{km})

$$w_\lambda = w_{\lambda_1-1} w_{\lambda_1+1, \lambda_{1,2}-1} \cdots w_{\lambda_{1,k-1}+1, \lambda_{1,k}-1} = w_{\lambda_1-1} w'.$$

Let $z = x z_1 y$, where x, y are in the subgroup of S_n generated by s_i , $i \neq \lambda_1$, and $l(s_i z_1) = l(z_1 s_i) = l(z_1) + 1$ if $i \neq \lambda_1$. Write $x = x_1 x_2$ and $y = y_1 y_2$, where x_1, y_1 are in the subgroup W_{λ_1-1} of S_n generated by $s_1, \dots, s_{\lambda_1-1}$ and x_2, y_2 are in the subgroup U_{λ_1+1} of S_n generated by $s_{\lambda_1+1}, \dots, s_{n-1}$.

We have $T_u C_\lambda = C_\lambda T_u = q^{l(u)} = q^{l(u)} C_\lambda$ for $u = x_1, y_1$ and $T_u C_{w_{\lambda_1-1}} = C_{w_{\lambda_1-1}} T_u$ for $u = x_2, y_2$. Note that $C_\lambda = C_{w_{\lambda_1-1}} C_{w'} = C_{w'} C_{w_{\lambda_1-1}}$. Thus

$$C_\lambda T_z C_\lambda = q^{l(x_1)+l(y_1)} C_{w'} T_{x_2} C_{w_{\lambda_1-1}} T_{z_1} C_{w_{\lambda_1-1}} T_{y_2} C_{w'}.$$

If $z_1 = e$, then

$$C_\lambda T_z C_\lambda = q^{l(x_1)+l(y_1)} [\lambda_1]! C_{w_{\lambda_1-1}} C_{w'} T_{x_2 y_2} C_{w'}.$$

We are reduced to the case $k = 1$.

Now assume that $z_1 \neq e$. By Lemma 1 we know that

$$C_{w_{\lambda_1-1}} T_{z_1} C_{w_{\lambda_1-1}} \in \pm q^* [\lambda_1 - j - 1] C_{w_{\lambda_1-1} w_{\lambda_1+1, \lambda_1+j}} + \sum_\nu F_\nu,$$

where $j \leq \lambda_1 - 1$ is defined by $z_1 = s_{\lambda_1} s_{\lambda_1-1} \cdots s_{\lambda_1-j} \cdots s_{\lambda_1+j} s_{\lambda_1+j-1} \cdots s_{\lambda_1}$, and ν runs through the partitions $(\lambda_1 + m, j + 1 - m, 1, \dots, 1)$, $j + 1 \geq m \geq 1$.

Note that both $C_\lambda T_z C_\lambda$ and $C_{w'} T_{x_2} C_{w_{\lambda_1-1} w_{\lambda_1+1, \lambda_1+j}} T_{y_2} C_{w'}$ are contained in $\sum_{\mu \geq \lambda} F_\mu$ and $C_{w'} T_{x_2} F_\nu T_{y_2} C_{w'} \subset \sum_{\mu \geq \nu} F_\mu$ for any ν . Whenever $\mu \geq \lambda$ and $\mu \geq (\lambda_1 + m, \dots)$ for some $m \geq 1$, we must have $\mu > \lambda$. Thus we have

$$C_\lambda T_z C_\lambda \in \pm q^* [\lambda_1 - j - 1]! C_{w'} T_{x_2} C_{w_{\lambda_1-1} w_{\lambda_1+1, \lambda_1+j}} T_{y_2} C_{w'} + \sum_{\mu > \lambda} F_\mu,$$

where $*$ stands for an integer. Let $\tau = (\lambda_1, j + 1, 1, \dots, 1)$. Then $w_{\lambda_1-1} w_{\lambda_1+1, \lambda_1+j} = w_\tau$. Note that $C_{w_{\lambda_1-1} w_{\lambda_1+1, \lambda_1+j}} = C_{\lambda_1-1} C_{w_{\lambda_1+1, \lambda_1+j}}$. If $j \geq \lambda_2$, then $\tau \not\leq \lambda$, so $C_{w'} T_{x_2} C_{w_{\lambda_1-1} w_{\lambda_1+1, \lambda_1+j}} T_{y_2} C_{w'}$ is contained in $(\sum_{\mu \geq \lambda} F_\mu) \cap (\sum_{\mu \geq \tau} F_\mu) \subset \sum_{\mu > \lambda} F_\mu$. We are done in this case.

Now assume that $j \leq \lambda_2 - 1$. Then $\lambda_1 - j - 1 \geq \lambda_1 - \lambda_2$ and $C_{w_{\lambda_1-1}} T_{z_1} C_{w_{\lambda_1-1}} \in [\lambda_1 - \lambda_2]! f_1 C_{w_{\lambda_1-1} w_{\lambda_1+1, \lambda_1+j}} + \sum_\nu F_\nu$ for some $f_1 \in K$, where ν runs through the partitions $(\lambda_1 + m, j + 1 - m, 1, \dots, 1)$, $j + 1 \geq m \geq 1$. Thus we have

$$C_\lambda T_z C_\lambda \in [\lambda_1 - \lambda_2]! f_1 C_{w_{\lambda_1-1}} C_{w'} T_{x_2} C_{w_{\lambda_1+1, \lambda_1+j}} T_{y_2} C_{w'} + \sum_\nu C_{w'} T_{x_2} F_\nu T_{y_2} C_{w'}.$$

Note that $x_2, w', y_2, w_{\lambda_1+1, \lambda_1+j}$ are all in the subgroup of S_n generated by s_i , $\lambda_1 + 1 \leq i \leq n - 1$ and $C_{w'} T_{x_2} F_\nu T_{y_2} C_{w'}$ is included in $\sum_{\mu \not\leq \lambda} F_\mu$ if $\nu = (\lambda_1 + m, j + 1 - m, 1, \dots, 1)$ for some $m \geq 1$. By induction hypothesis, we see the lemma is true.

Lemma 4. Let λ be as in Lemma 3. Set

$$z_i = (s_{\lambda_{1i}} s_{\lambda_{1i}-1} \cdots s_{\lambda_{1i}-\lambda_{i+1}+1}) \cdots (s_{\lambda_{1,i+1}-1} s_{\lambda_{1,i+1}-2} \cdots s_{\lambda_{1,i}}),$$

for $i = 1, 2, \dots, k - 1$. Define

$$h = T_{z_{k-1}} (T_{z_{k-2}} T_{z_{k-1}}) (T_{z_{k-3}} T_{z_{k-2}} T_{z_{k-1}}) \cdots (T_{z_1} T_{z_2} \cdots T_{z_{k-1}}).$$

Then $C_\lambda h C_\lambda \in \pm q^* \prod_{i=1}^k ([\lambda_i - \lambda_{i+1}]!)^i C_\lambda + F_{>\lambda}$, where $*$ stands for an integer and $F_\lambda = \sum_{\mu > \lambda} F_\mu$.

Proof: Set $u_i = C_{w_{\lambda_{1,i-1}+1, \lambda_{1i}-1}}$ (we understand that $\lambda_{1,0} = 0$) and $h_i = T_{z_i}$. Then $C_\lambda = u_1 u_2 \cdots u_k$, $u_i u_j = u_j u_i$ for all i, j , and $u_i h_j = h_j u_i$ if $i < j$. For $h', h'' \in H$ and $F \subset H$, we write $h' \equiv h'' + F$ if $h' - h'' \in F$.

Using Lemma 1 we get

$$\begin{aligned}
C_\lambda h C_\lambda &= u_k(u_{k-1}h_{k-1})(u_{k-2}h_{k-2}h_{k-1}) \cdots \\
&\quad \times (u_2h_2h_3 \cdots h_{k-1})u_1h_1u_1h_2u_2 \cdots h_{k-1}u_{k-1}u_k \\
&\equiv \pm q^*[\lambda_1 - \lambda_2]! u_k(u_{k-1}h_{k-1})(u_{k-2}h_{k-2}h_{k-1}) \cdots \\
&\quad \times (u_2h_2h_3 \cdots h_{k-1})u_1u_2h_2u_2 \cdots h_{k-1}u_{k-1}u_k + F_{>\lambda} \\
&\equiv \pm q^*[\lambda_1 - \lambda_2]! u_1u_k(u_{k-1}h_{k-1})(u_{k-2}h_{k-2}h_{k-1}) \cdots \\
&\quad \times (u_2h_2h_3 \cdots h_{k-1})u_2h_2u_2 \cdots h_{k-1}u_{k-1}u_k + F_{>\lambda} \\
&\equiv \pm q^*[\lambda_1 - \lambda_2]! [\lambda_2 - \lambda_3]! u_1u_k(u_{k-1}h_{k-1})(u_{k-2}h_{k-2}h_{k-1}) \cdots \\
&\quad \times (u_2h_2h_3 \cdots h_{k-1})u_2u_3h_3u_3 \cdots h_{k-1}u_{k-1}u_k + F_{>\lambda} \\
&\equiv \pm q^*[\lambda_1 - \lambda_2]! ([\lambda_2 - \lambda_3]!)^2 u_1u_2u_k(u_{k-1}h_{k-1})(u_{k-2}h_{k-2}h_{k-1}) \cdots \\
&\quad \times (u_3h_3 \cdots h_{k-1})^2 u_3h_3u_3 \cdots h_{k-1}u_{k-1}u_k + F_{>\lambda} \\
&\equiv \cdots \\
&\equiv \pm q^* \prod_{i=1}^k ([\lambda_i - \lambda_{i+1}]!)^i C_\lambda + F_{>\lambda}.
\end{aligned}$$

Combining Lemmas 3 and 4 we see that part (b) of the theorem is true. The theorem is proved.

If $\sum_{w \in S_n} q^{l(w)} \neq 0$ and K is an algebraic closed field of characteristic 0, then we have the Deligne-Langlands-Lusztig classification for irreducible modules of \tilde{H} (see [BZ, Z], [KL1], [X]). We have another classification due to Ariki and Mathas for any sufficient large K (see [AM]). An interesting question is to classify irreducible modules of \tilde{H} in the spirit of Deligne-Langlands-Lusztig classification when $\sum_{w \in S_n} q^{l(n)} = 0$, see [Gr] for an announcement. If one can manage the calculation $C_\lambda \tilde{H} C_\lambda$ to get counterparts of Lemmas 3 and 4, the question will be settled.

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